# Evolution of mixing during the damping of a number state

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**Abstract.** We carry out a detailed analysis of the mixing process during the interaction of a single-mode field initially in a number state with a thermal reservoir. We prove that a maximum in the evolution of the 2-entropy of the attenuated mode exists provided that its initial photon number exceeds the mean occupancy of the reservoir. This transient mixing enhancement can be considered as a quantum effect of the initial state on the mode damping.

## 1. Introduction

The decay of a single-mode radiation field in a dissipative environment has been studied, in particular, in connection with the decoherence of the system [1–3]. Zurek *et al* have used the master equation of the quantum Brownian motion to model the interaction with an environment and have found the coherent states to have the least increase in entropy (maximal states). In their treatment the coherent states are thus the most *classical* ones [1]. Most recently, Părăoanu and Scutaru [3, 4] gave a comprehensive discussion of decoherence as a mechanism for the selection of maximal states within the framework of the Lindblad equation formalism [5]. In the case of Gaussian states they have found that, in general, the maximal ones are determined by the diffusion coefficients of the environment. Note that, apart from the advantage of describing many interesting phenomena, the master equations of Lindblad type have the essential property of preserving the positivity of the density operator. As a consequence, the Robertson–Schrödinger uncertainty relation holds at any time [6]. The quantum optical master equation [7], which is of the Lindblad type, has been frequently used to study the destruction of non-classical properties of a field state by thermal noise [8–10]. For example, in [10] we have found a limit time at which all squeezing effects disappear by decoherence, regardless of the initial state of the mode. On the other hand, recent works report that, in special cases, the environment enhances non-classical properties. In [10–12], a single-mode field in a superposition of two coherent states weakly coupled to a heat bath at zero [11] or very low temperature [10, 12] was studied. It was found that fourth-order squeezing [11, 12] and, more generally, squeezing to any order [10] could be created transiently due to the interaction with the environment.

Similarly to our earlier work [10], here we make use of the quantum optical master equation in order to investigate the evolution of mixing when a one-mode field initially in a non-classical state is put in contact with a thermal bath. An efficient tool to analyse the mixing produced by the environment is the 2-entropy of the damped mode. In [10] we found very

interesting features of the mixing when the input field is in an even coherent state. We now show that apparently these features are common to other non-classical states. We have chosen to analyse the behaviour of the 2-entropy of a fundamental quantum state, the number state, when admixed with thermal noise.

This paper is organized as follows. In section 2 we express the solution of the master equation in terms of the characteristic function (CF) of the initial field state. The 2-entropy of a damped Fock state is evaluated and discussed in section 3. From the analysis therein, we distinguish a classical regime of mixing from a non-classical one by comparing the initial mean photon number of the mode with the mean occupancy of the reservoir. In section 4 we point out the loss of non-classicality by means of the $P$ representation of the density operator. Section 5 summarizes our conclusions.

## 2. Characteristic function of the damped mode

We deal with a single-mode radiation field of angular frequency $\omega$ whose amplitude operators are denoted by $a$ and $a^\dagger$. For any state of the field the CF [13] defined as the expectation value of the Weyl displacement operator $D(\lambda) = \exp(\lambda a^\dagger - \lambda^* a)$,

$$\chi(\lambda) = \mathrm{Tr}[\rho D(\lambda)] \tag{2.1}$$

determines uniquely the density operator $\rho$. Indeed, the CF is the weight function in the Weyl expansion [14] of the density operator,

$$\rho = \frac{1}{\pi} \int \mathrm{d}^2\lambda \, \chi(\lambda) D(-\lambda). \tag{2.2}$$

The normally ordered CF,

$$\chi^{(\mathcal{N})}(\lambda) = \mathrm{Tr}(\rho e^{\lambda a^\dagger} e^{-\lambda^* a}) \tag{2.3}$$

has the Taylor expansion

$$\chi^{(\mathcal{N})}(\lambda) = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{1}{l!m!} \lambda^l (-\lambda^*)^m \langle (a^\dagger)^l a^m \rangle \tag{2.4}$$

from which one readily finds the expectation values $\langle (a^\dagger)^l a^m \rangle$ which are necessary when examining the statistical properties of the state. Note also that the Fourier transform of the normally ordered CF is the Glauber–Sudarshan $P$ representation [15],

$$P(\beta) = \frac{1}{\pi} \int \mathrm{d}^2\lambda \exp(\beta\lambda^* - \beta^*\lambda) \chi^{(\mathcal{N})}(\lambda). \tag{2.5}$$

We now consider a weak coupling of the field mode with a thermal reservoir whose mean occupancy is $\bar{n}_R$. In the interaction picture, the evolution of the reduced density operator of the field is described by the quantum optical master equation [7],

$$\frac{\partial \rho}{\partial t} = \tfrac{1}{2}\gamma(2a\rho a^\dagger - a^\dagger a\rho - \rho a^\dagger a) + \gamma\bar{n}_R(a^\dagger\rho a + a\rho a^\dagger - a^\dagger a\rho - \rho a a^\dagger) \tag{2.6}$$

with $\gamma$ the field-reservoir coupling constant. Equation (2.6) is derived in [7] on physical grounds: the field is sustained by a cavity which stands for the environment. The effective interaction between the mode and the cavity consists in photon scattering into and out of the mode. For weak coupling this leads to the rotating-wave approximation. By adopting the Markov approximation to the second order in the interaction, it is recognized that the reservoir

correlation time is negligible on a time scale $\sim 1/\gamma$ in which the mode loses significant energy. Equation (2.6) allows for the diffusion of fluctuations of the reservoir into the system mode (the $\gamma \bar{n}_R$ terms).

Our method in treating the evolution of the damped field is to use its CF via the Weyl expansion (2.2). This expansion is valid at any time due to the remarkable properties of the master equations of the Lindblad type [6]. To this end, we convert the master equation (2.6) into a first-order partial differential equation for the normally ordered CF. We find

$$\frac{\partial \chi^{(\mathcal{N})}}{\partial t} = -\gamma \bar{n}_R |\lambda|^2 \chi^{(\mathcal{N})} - \left(\tfrac{1}{2}\gamma - i\omega\right)\lambda \frac{\partial \chi^{(\mathcal{N})}}{\partial \lambda} - \left(\tfrac{1}{2}\gamma + i\omega\right)\lambda^* \frac{\partial \chi^{(\mathcal{N})}}{\partial \lambda^*}. \quad (2.7)$$

The solution of this equation, given by the characteristic-curve method [16], depends on its initial form $\chi^{(\mathcal{N})}(\lambda, 0)$ as

$$\chi^{(\mathcal{N})}(\lambda, t) = \chi^{(\mathcal{N})}(\lambda e^{-(\gamma/2 - i\omega)t}, 0) \exp\left[-\bar{n}_R(1 - e^{-\gamma t})|\lambda|^2\right]. \quad (2.8)$$

In contrast to the CF, which is picture independent, the diagonal $P$ representation is not. It is easy to show that, in the interaction picture, one has to use in the Fourier integral (2.5) a modified form of the CF from which the oscillatory factors were removed. For a more detailed account of this issue see [9], section 2.

The mean photon number of the damped state is obtained from the normally ordered CF (2.8) as

$$\bar{n}(t) = \bar{n}(0)\, e^{-\gamma t} + \bar{n}_T(t) \quad (2.9)$$

where

$$\bar{n}_T(t) := \bar{n}_R[1 - \exp(-\gamma t)] \quad (2.10)$$

is the thermal mean occupancy in the field mode at time $t$. The significance of the factorization (2.8) is quite clear: it describes the superposition of the attenuated field with a thermal one whose time-dependent mean occupancy is $\bar{n}_T(t)$, equation (2.10). Therefore, the decay of the field mode ruled by the quantum optical master equation is a *thermalization process*.

## 3. Mixing by damping

It is convenient to employ as a measure of the degree of mixing the 2-entropy

$$S_2(\rho) := -\ln[\mathrm{Tr}(\rho^2)]. \quad (3.1)$$

Equation (3.1) is the special case $\tilde{\alpha} = 2$ of the quantum-mechanical counterpart of a Rényi $\tilde{\alpha}$-entropy [17],

$$S_{\tilde{\alpha}}(\rho) := \frac{1}{1 - \tilde{\alpha}} \ln[\mathrm{Tr}(\rho^{\tilde{\alpha}})] \quad (\tilde{\alpha} > 0). \quad (3.2)$$

Now we focus on the mixing process described by the master equation (2.6) when the initial state is a Fock state with the photon number denoted by $M$. The CF of a pure number state is the corresponding diagonal matrix element of the displacement operator in the Fock basis. We recall its expression,

$$\chi(\lambda) = \langle M | D(\lambda) | M \rangle$$
$$= \exp\left(-\tfrac{1}{2}|\lambda|^2\right) L_M(|\lambda|^2) \quad (3.3)$$

where $L_M(x)$ denotes a Laguerre polynomial. According to equation (2.8), the CF of the damped number state is

$$\chi(\lambda, t) = \exp\left\{-|\lambda|^2\left[\bar{n}_T(t) + \tfrac{1}{2}\right]\right\}L_M\left(|\lambda|^2 e^{-\gamma t}\right). \tag{3.4}$$

The Weyl expansion (2.2) yields the time-dependent degree of purity as the integral

$$\mathrm{Tr}\{[\rho(t)]^2\} = \frac{1}{\pi}\int d^2\lambda\,|\chi(\lambda, t)|^2. \tag{3.5}$$

In the case of a damped Fock state, the relation between a Laguerre polynomial and the confluent hypergeometric function $_1F_1$,

$$L_l^{(\mu)}(x) = \frac{(\mu + 1)_l}{l!}\,_1F_1(-l; \mu + 1; x) \tag{3.6}$$

allows one to take advantage of the well known integral [18]

$$\int_0^\infty dt\,\exp(-st)t^{c-1}\,_1F_1(a; c; \sigma t)\,_1F_1(a'; c; \sigma't) = \Gamma(c)\,s^{a+a'-c}(s - \sigma)^{-a}(s - \sigma')^{-a'}$$

$$\times\,_2F_1\left(a, a'; c; \sigma\sigma'(s - \sigma)^{-1}(s - \sigma')^{-1}\right). \tag{3.7}$$

In equation (3.7), $_2F_1$ is a Gauss hypergeometric function. The result is

$$\mathrm{Tr}\{[\rho(t)]^2\} = \frac{[2\bar{n}_T(t) + 1 - e^{-\gamma t}]^{2M}}{[2\bar{n}_T(t) + 1]^{2M+1}}\,_2F_1\left(-M, -M; 1; \frac{e^{-2\gamma t}}{[2\bar{n}_T(t) + 1 - e^{-\gamma t}]^2}\right). \tag{3.8}$$

To obtain a more suitable analytic form of the Gauss function, we apply the transformation formula [18]

$$_2F_1(a, b; a - b + 1; z) = (1 + \sqrt{z})^{-2a}\,_2F_1\left(a, a - b + \tfrac{1}{2}; 1; \frac{4\sqrt{z}}{(1 + \sqrt{z})^2}\right) \tag{3.9}$$

and find

$$\mathrm{Tr}\{[\rho(t)]^2\} = \frac{1}{2\bar{n}_T(t) + 1}\,_2F_1\left(-M, \tfrac{1}{2}; 1; v(t)\right). \tag{3.10}$$

The argument of the Gauss hypergeometric function in equation (3.10) is

$$v(t) := 1 - [\eta(t)]^2 \tag{3.11}$$

where $\eta(t)$ is a convenient dimensionless variable determined by the reservoir only:

$$\eta(t) := 1 - \frac{2e^{-\gamma t}}{2\bar{n}_T(t) + 1}. \tag{3.12}$$

Note that $\eta(t)$ is a strictly increasing and concave function of time which varies from $\eta(0) = -1$ to $\eta(\infty) = 1$, having its zero at the time

$$t_m = \frac{1}{\gamma}\ln\left(1 + \frac{1}{2\bar{n}_R + 1}\right). \tag{3.13}$$

At the same point $t_m$ the function $v(t)$ has a unique maximum $v(t_m) = 1$. According to equations (3.1) and (3.10), the 2-entropy for a damped Fock state is

$$S_2[\rho(t)] = \ln[2\bar{n}_T(t) + 1] - \ln\left[_2F_1\left(-M, \tfrac{1}{2}; 1; v(t)\right)\right]. \tag{3.14}$$

Now, we analyse the evolution of the 2-entropy by using its first-order time derivative,

$$\frac{\partial S_2[\rho(t)]}{\partial t} = \gamma(1 - \eta)w(\eta). \tag{3.15}$$

In equation (3.15) we have introduced the function

$$w(\eta) := \bar{n}_R - M\eta[1 + \bar{n}_R(1 - \eta)]\frac{{}_2F_1\left(-M + 1, \frac{3}{2}; 2; 1 - \eta^2\right)}{{}_2F_1\left(-M, \frac{1}{2}; 1; 1 - \eta^2\right)}. \tag{3.16}$$

The initial value of the slope (3.15),

$$\Gamma_m := \left.\frac{\partial S_2}{\partial t}\right|_{t=0} \tag{3.17}$$

can be defined as a conventional *rate of mixing*, since it is a measure of the rapidity with which the initial state loses its purity. Note that the rate of mixing is

$$\Gamma_m = 2\gamma[\bar{n}_R + (2\bar{n}_R + 1)M] \tag{3.18}$$

and increases with the initial photon number $M$ in the mode, as well as with the mean occupancy of the reservoir $\bar{n}_R$. At the time $t_m$, the slope (3.15) reaches a value independent of $M$,

$$\left.\frac{\partial S_2}{\partial t}\right|_{t=t_m} = \gamma\bar{n}_R. \tag{3.19}$$

An important point of our discussion is the sign of the derivative (3.15) for large times. Let us note its asymptotic limit

$$\lim_{t\to\infty}\frac{\partial S_2}{\partial t} = \gamma(\bar{n}_R - M)\lim_{\eta\to 1-0}(1 - \eta). \tag{3.20}$$

When $\bar{n}_R - M > 0$, we have succeeded to prove that the function $w(\eta)$ is positive for every $\eta$. Consequently, the 2-entropy is then a strictly increasing function of time. This regime of mixing may be called *classical* because such a behaviour of the 2-entropy is typical for a coherent (quasi-classical) state [9, 10].

In contrast, when $\bar{n}_R - M < 0$, the 2-entropy has at least a maximum because its first-order time derivative, equation (3.15), changes its sign between $\eta = 0$ and 1. In [10] the regime of mixing characterized by a maximum in the evolution of the 2-entropy has been termed *non-classical*. For $\bar{n}_R > 0$, owing to the rather complicated transcendental nature of the function (3.14), a maximum of the 2-entropy, when existing, cannot be found by analytic means. However, a lower bound of the maximal entropy is the value

$$S_2[\rho(t_m)] = \ln\left(\frac{2\bar{n}_R + 1}{\bar{n}_R + 1}\right) + \ln\left[\frac{M!2^M}{(2M - 1)!!}\right] \tag{3.21}$$

which provides a measure of the degree of mixing. This is the reason for having denominated $t_m$ in [10] as the *mixing time*.

The most typical example of a non-classical regime of mixing is afforded by the damping in contact with a zero-temperature bath ($\bar{n}_R = 0$). We are justified in calling *dissipation* this kind of damping. Equations (3.17) and (3.13) show that the 2-entropy reaches a unique maximum precisely at the mixing time
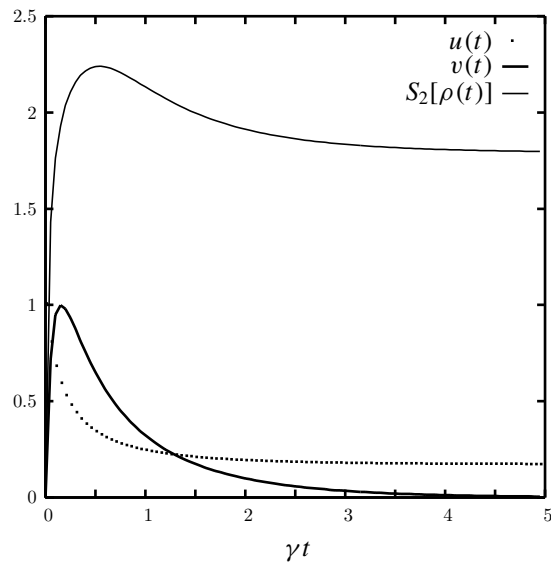
$$t_m = \frac{1}{\gamma}\ln 2. \tag{3.22}$$

According to equation (3.21), the maximal entropy

$$S_2[\rho(t_m)] = \ln\left[\frac{M!2^M}{(2M - 1)!!}\right] \tag{3.23}$$
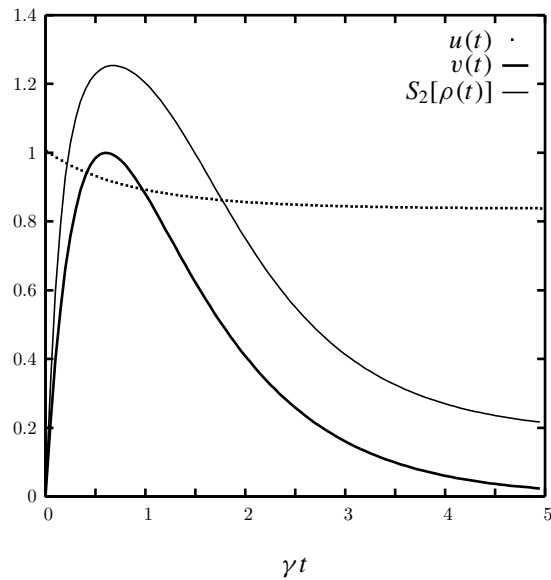
**Figure 1.** Time evolution of the functions $u(t)$, $v(t)$ and $S_2[\rho(t)]$ when the mean occupancy of the bath is $\bar{n}_R = 0.1$ and $M = 3$. The bath is characterized by $\gamma t_m = 0.606$ and $\gamma t_c = 2.398$.



**Figure 2.** As in figure 1, but for $\bar{n}_R = 2.5$ and $M = 5$. We have here $\gamma t_m = 0.154$ and $\gamma t_c = 0.336$.

increases with the initial photon number $M$. To resume, dissipation of the mode specifically includes for $t < t_m$ a mixing of the initial Fock state with the rate $\Gamma_m = 2\gamma M$, followed at times $t > t_m$ by a complete demixing ending in the vacuum state.

In figure 1 we have plotted the functions $u(t) := 1/(2\bar{n}_T(t) + 1)$, $v(t)$ and $S_2[\rho(t)]$, for $\bar{n}_R = 0.1$ ($\gamma t_m = 0.606$) and $M = 3$. Note that $u(t)$ decreases from $u(0) = 1$ to $u(\infty) = 1/(2\bar{n}_R + 1)$. For small values of $\bar{n}_R$, as in figure 1, the evolution of the 2-entropy is
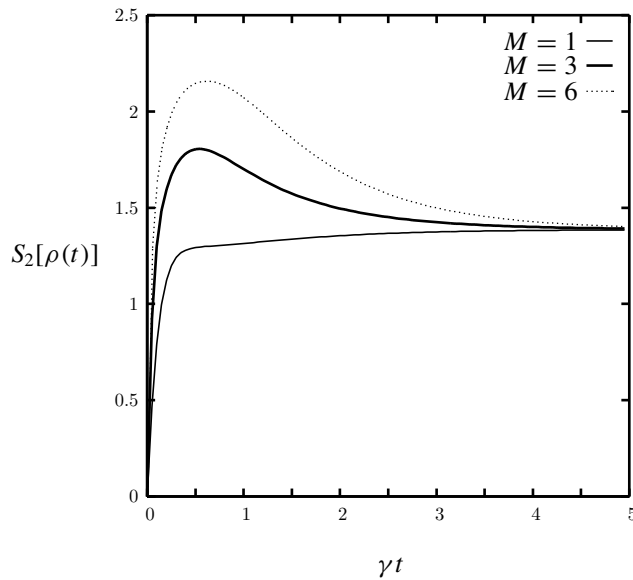
**Figure 3.** 2-entropy production governed by the master equation (2.6) for an input Fock state plotted for several values of $M$ and for $\bar{n}_R = 1.5$. Note that in this case $\gamma t_m = 0.223$; $\gamma t_c = 0.511$.

close to that displayed in the dissipation case. In particular, the time $t_m$ at which $v(t)$ reaches its maximum is a good lower approximation for the time $t_{max}$ corresponding to the maximum of the 2-entropy. The situation changes for larger values of $\bar{n}_R$ when $t_m$ decreases and the difference $t_{max} - t_m$ increases in agreement with equation (3.19). This case is illustrated in figure 2 ($M = 5$, $\bar{n}_R = 2.5$, $\gamma t_m = 0.154$).

In figure 3 we have plotted 2-entropy versus $\gamma t$ for several values of $M$ when $\bar{n}_R = 1.5$ ($\gamma t_m = 0.223$). The graphs exhibit a monotonic increase of 2-entropy for $\bar{n}_R - M > 0$ (case $M = 1$, *classical regime*) and a unique maximum for $\bar{n}_R - M < 0$ (cases $M = 3, 6$, *non-classical regime*).

## 4. $P$ representation

In the interaction picture, the $P$ representation is the Fourier transform of the non-oscillating normally ordered CF:

$$\chi^{(\mathcal{N}\mathcal{I})}(\lambda, t) := \chi^{(\mathcal{N})}\left(\lambda \exp\left(-\tfrac{1}{2}\gamma t\right), 0\right) \exp\left[-\bar{n}_T(t)|\lambda|^2\right]. \tag{4.1}$$

When using equations (4.1) and (3.4) in equation (2.5) we are left to evaluate the integral

$$P(\beta, t) = \frac{1}{\pi} \int d^2\lambda \, \exp[\beta\lambda^* - \beta^*\lambda - \bar{n}_T(t)|\lambda|^2] L_M(|\lambda|^2 e^{-\gamma t}). \tag{4.2}$$

This can be easily done in an indirect way. We first perform a similar integral in which $L_M(x)$ is replaced by the generating function of the Laguerre polynomials [19]

$$(1 - z)^{-1} \exp\left(\frac{xz}{z - 1}\right) = \sum_{M=0}^{\infty} L_M(x) z^M \qquad |z| < 1. \tag{4.3}$$

The formula applied to this end,

$$\int d^2\lambda \, \exp(\beta\lambda^* - \beta^*\lambda - K|\lambda|^2) = \frac{\pi}{K} \exp\left(-\frac{1}{K}|\beta|^2\right) \qquad K > 0 \tag{4.4}$$

is a particular case of the Gaussian integral (A6)–(A8) from our paper [20]. In both sides of the resulting identity use is then made of the expansion (4.3). By equalling the coefficients of $z^M$ on both sides, we finally establish the explicit formula

$$P(\beta, t) = \frac{1}{\bar{n}_T(t)} \left[ \frac{\bar{n}_T(t) - e^{-\gamma t}}{\bar{n}_T(t)} \right]^M \exp\left[ -\frac{|\beta|^2}{\bar{n}_T(t)} \right] L_M \left( -\frac{|\beta|^2 e^{-\gamma t}}{\bar{n}_T(t)[\bar{n}_T(t) - e^{-\gamma t}]} \right). \qquad (4.5)$$

If the argument of the Laguerre polynomial in equation (4.5) is negative, the $P$ representation is positive for any $\beta$. This implies a time condition which is *sufficient* for the existence of the $P$ representation as a well behaved function:

$$\bar{n}_T(t) - e^{-\gamma t} \geqslant 0. \qquad (4.6)$$

At the limit time

$$t_c := \frac{1}{\gamma} \ln\left( 1 + \frac{1}{\bar{n}_R} \right) \qquad (4.7)$$

all the non-classical properties of the initial Fock state disappear due to the interaction with the heat bath. Obviously, $t_c > t_m$. Note that the non-classicality time $t_c$ becomes infinite in the extreme case of dissipation ($\bar{n}_R = 0$).

## 5. Conclusions

The existence of a maximum of the 2-entropy, depending on the initial non-classical state, is undoubtedly a quantum effect emphasized formerly for Gaussian states [9] and more recently for an even coherent state [10]. The description of the mixing process introduced in [10] for a field mode in an even coherent state in contact with a thermal reservoir is also appropriate for a damped Fock state. To sum up, we have recovered here quantities and conditions that have the same expressions or significance as in [10]:

(a) the mixing time, equation (3.13);
(b) the rate of mixing which can be written in a more general form

$$\Gamma_m = 2\gamma[\bar{n}_R + (2\bar{n}_R + 1)\bar{n}(0)] \qquad (5.1)$$

where $\bar{n}(0)$ is the initial mean photon number of the field;
(c) the slope of the 2-entropy at the time $t_m$, equation (3.19);
(d) the two regimes of mixing defined by the relation between the mean photon number in the initial mode and the mean occupancy of the bath;
(e) the time of mixing by dissipation, equation (3.22);
(f) the non-classicality time $t_c$, equation (4.7).

These properties, found in two important particular cases by explicit calculations, are yet to be derived for an arbitrary input state. Work along these lines is in progress.

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